Abstract CR structures
Lecture 13

<u>Recall</u>. A <u>CR structure</u> on a smooth mfld M is a complex subbundle $\mathcal{V}$ ($= T^{1,0}M$) of $\mathbb{C}TM$ s.t.

(i) $\mathcal{V}_p \cap \overline{\mathcal{V}_p} = \{0\}$, $\forall p \in M$

(ii) "$[\mathcal{V}, \mathcal{V}] \subseteq \mathcal{V}$", i.e. $\forall$ sections Z, W of $\mathcal{V}$, $[Z, W]$ is a section of $\mathcal{V}$.

- CR dim $M = \dim_{\mathbb{C}} \mathcal{V}$. If $m = \dim_{\mathbb{R}} M$, $k =$ CR dim M, then $\dim_{\mathbb{C}} \mathbb{C}T_pM = m$ and $\dim_{\mathbb{C}} \mathcal{V} \oplus \overline{\mathcal{V}} = 2k$. The difference ($= \dim_{\mathbb{C}} \mathbb{C}T_pM / \mathcal{V} \oplus \overline{\mathcal{V}}$) $d' = m - 2k$ is called <u>CR codimension</u> of M.

- Note that $\mathcal{V} \oplus \overline{\mathcal{V}}$ is <u>real</u>, i.e. $X_p \in \mathcal{V} \oplus \overline{\mathcal{V}} \Rightarrow \overline{X_p} \in \mathcal{V} \oplus \overline{\mathcal{V}}$. Thus, $H = (\mathcal{V} \oplus \overline{\mathcal{V}}) \cap TM$ is real subbundle of TM (of $\dim_{\mathbb{R}} H = 2k$) and

$$\mathbb{C} \otimes H = \mathcal{V} \oplus \overline{\mathcal{V}}.$$

We conclude that, locally, we can find real vector fields $T_1, \dots, T_{d'}$ ($d' = m - 2k$ as above) s.t. $H \oplus \underbrace{\mathbb{R}\{T_1, \dots, T_{d'}\}}_{\text{Span}/\mathbb{R} \text{ of } T_j} = TM$.

Or, equiv.,

- $\mathcal{V} \oplus \overline{\mathcal{V}} \oplus \mathbb{C}\{T_1, \dots, T_{d'}\} = \mathbb{C}TM$.

- The pair $(M, \mathcal{V})$ (or simply M if $\mathcal{V}$ is understood) is called a <u>CR manifold</u>

- A CR structure of CR codim $= 0$ (i.e. s.t. $H = TM$ or $\mathcal{V} \oplus \overline{\mathcal{V}} = \mathbb{C}TM$) is called <u>almost complex</u>. The <u>Newlander–Nirenberg Thm</u> (above) states: Every smooth almost complex structure is complex (i.e. comes from a cplx mfld).

A pf of N-N can be found in Hörmander Sec. 5.7

Formal integrability vs. integrability.

Frobenius Thm. Let $U \subseteq \mathbb{R}^m$ be open and $X_1, \ldots, X_n$ real vector fields in U. If $X_1, \ldots, X_n$ are linearly indep. and

$$(*) \qquad [X_i, X_j] = \sum_k a_{ijk} X_k, \quad i,j = 1, \ldots, n.$$

then $\exists$ coordinates $(x,y) = (x_1, \ldots, x_n, y_1, \ldots, y_{m-n})$ near each $p_0 \in U$ s.t.

$$X_j = \sum_{k=1}^{n} c_{jk} \frac{\partial}{\partial x_j}, \quad j = 1, \ldots, n.$$

See [BER] for references to proof.

Note: Condition $(*)$ is called the integrability condition and is formally the same as the formal integrability (ii) in CR structure def. The fact that (ii) is for complex vector fields makes a significant difference and the conclusion of FT does not hold for CR structures.

Minimality and Finite Type

The conclusion in FT $\Rightarrow$ In the local coordinates $x=(x_1,\dots,x_n,\dots,x_m)$ the n-$\dim_{\mathbb{R}}$ mflds $\Pi_{y_0}=\{(x,y)\in\mathbb{R}^n\times\mathbb{R}^{m-n} : y=y_0\}$ satisfy $X_1,\dots,X_n$ form a basis for their tangent bundles $T\Pi_{y_0}$, $\forall y_0 \sim 0$.

Such a conclusion <u>does not</u> hold for a CR mfld $(M,\mathcal{V})$. In fact, in general, $\nexists\, M'\subseteq M$ st. $\dim_{\mathbb{R}} M' < \dim_{\mathbb{R}} M$ and $\mathcal{V}\subseteq \mathbb{C}TM'$. If such M' exists, $(M',\mathcal{V})$ is a CR mfld w/ CRdim M' = CRdim M.

<u>Def.</u> $(M,\mathcal{V})$ is called <u>minimal</u> at $p\in M$ if <u>no</u> such $(M',\mathcal{V})$ exists with $p\in M'$.

How does FT fail for complex v.f.s?

If $(M, \mathcal{V})$ is a CR mfld of $\dim_{\mathbb{R}} M = m$ and CR dim $M = n$ and $Z_1, \dots, Z_n$ is a local frame for $\mathcal{V}$ near some $p_0 \in M$, then $X_{2j} = \frac{1}{2}(Z_j + \overline{Z}_j)$, $X_{2j+1} = \frac{1}{2i}(Z_j - \overline{Z}_j)$ are real vector fields and they span $\mathcal{H} = (\mathcal{V} \oplus \overline{\mathcal{V}}) \cap TM$. The integrability (*) in FT consists of considering, e.g.,

$$[X_{2i}, X_{2j+1}] = \frac{1}{4i}\Big([Z_i, Z_j] - [\overline{Z}_i, \overline{Z}_j] - [Z_i, \overline{Z}_j] + [\overline{Z}_i, Z_j] \Big).$$

The formal integrability (ii) $\Rightarrow$ $[Z_i, Z_j]$ is a section of $\mathcal{V}$ and $[\overline{Z}_i, \overline{Z}_j]$ a section of $\overline{\mathcal{V}}$. Their difference $\times \frac{1}{4i}$ is real and, hence, belongs to $\mathcal{H}$, <u>but</u> (ii) makes no requirement about $[Z_i, \overline{Z}_j]$ and $[\overline{Z}_i, Z_j]$.

If $M' \subseteq M$ and $\mathcal{V} \subseteq \mathbb{C}TM'$ ($\Leftrightarrow$ $H \subseteq TM'$), then $[X, X']$ must be still tangent to M' for any sections X, X' of H $\Leftrightarrow$ $[Z, \bar{Z}']$ is tangent to M' for all sections Z, Z' of $\mathcal{V}$. Same is true for repeated commutators $[\cdot\cdot[X, X'], \dots X^{(l-1)}]$.

<u>Def.</u> M is of <u>finite (commutator/Hörmander) type</u> at $p \in M$ if the collection of $[\cdot\cdot[Y^1, Y^2], \dots Y^l]$, where $Y^1, Y^2, \dots, Y^l$ range over all sections of H near p, spans T_pM. The smallest l for which all such commutators span is called <u>the type</u> of M at p (if M is of finite type at p).